Analysis of Floating-Point Matrix Multiplication Computed via Integer Arithmetic

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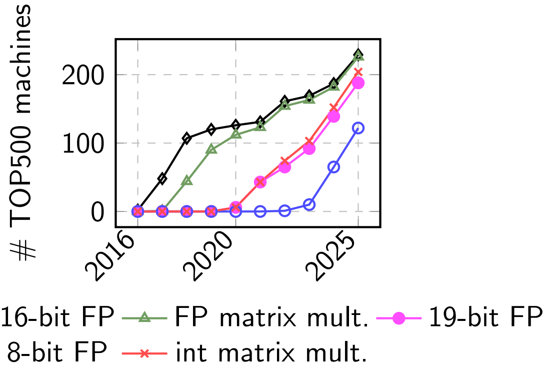
Fast low-precision matrix mult on the TOP500



Devices counted: P100, V100, A100, H100, MI210, MI250X, MI300X, Intel Data Center GPU, from `https://www.top500.org`.

NVIDIA HGX B200 throughputs (OPS/s)
int8 (4.5×10^{15}) fp16 (2.2×10^{15}) fp64 (0.04×10^{15}).

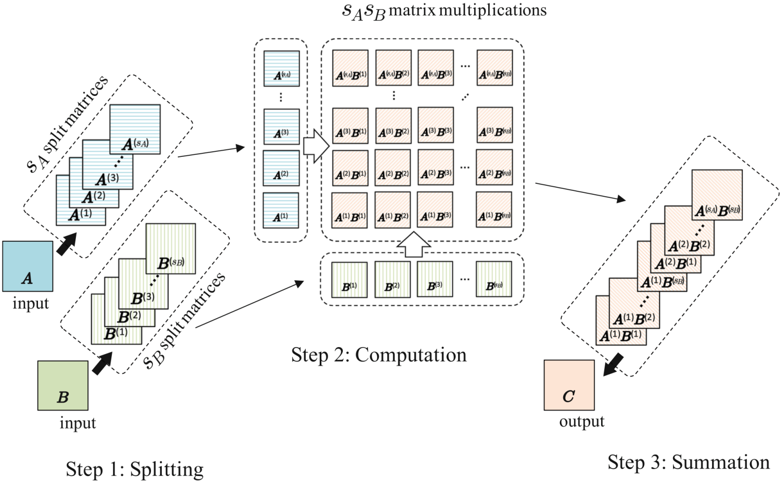
What is the "Ozaki scheme"?

In a nutshell

Split matrices into several "slices", compute many exact matrix multiplies, sum the products → more accurate than one $m \times k \times n$ product.

- [Ozaki, Ogita, Oishi, Rump, 2012] accurate fp64 mat mul by splitting the inputs and performing several exact fp64 mat mul + sum: $AB = \sum \mathrm{fl}(A^{(r)}B^{(s)})$.
- [Ozaki, Ogita, Oishi, Rump, 2013] improve the algorithm by exploiting *last non-zero bit* to reduce the number of splits/products.
- [Mukunoki, Ozaki, Ogita, Imamura, 2020] used 16-bit input, 32-bit output FP tensor cores for the above.
- [Ootomo, Ozaki, Yokota, 2024] used 8-bit input, 32-bit output **integer** tensor cores.
- [Mukunoki, 2026] uses an 8-bit FP tensor core.
- [Schwarz et al., 2026] tackles optimal slicing to 8 bit INT.

Mukunoki et al. 2020 illustration of the Ozaki scheme



FP64 $\to$ int8-int32 tensor core $\to$ FP64

Using integer tensor cores

A hardware FP64 FMA is multiplying 53-bit integers and adding 106-bit integers—FP64 fundamentally is computed through integers.

Similarly, **we can use integer tensor cores for FP64 mat-mul computations**.

Take $A \in \mathbb{F}^{n \times k}$ and $B \in \mathbb{F}^{k \times n}$. Goal: approximate AB by integer tensor cores. Typically $\mathbb{F}$ is double prec.

Assume A and B have no infinities, NaNs, or -0 and the computation cannot produce inf and NaN.

FP64 $\rightarrow$ int8-int32 tensor core $\rightarrow$ FP64

Step 1: Scale factors

1. For each row i of A, compute a scale factor

$$\alpha_i = 2^{\lfloor \log_2 M_i \rfloor} \times 2, \qquad M_i = \max_{1 \le j \le k} |a_{ij}|, \qquad 1 \le i \le m,$$

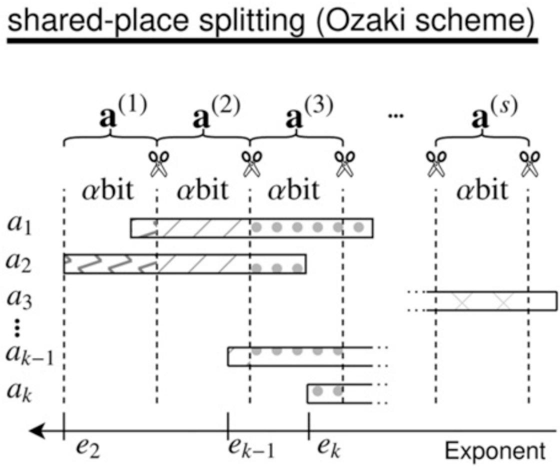
2. This guarantees each element scaled down by α_i is in $[0, 1)$.
3. And $0.5 \le M_i / \alpha_i < 1$.
4. Similar for B, **column-wise**.

Block floating-point arithmetic

Note that this is block floating-point representation of [Wilkinson, 1963]. **Ozaki scheme: each row of A and colum of B is a block with its own scale factor.**

Utilising 8-bit integer tensor cores [Ootomo et al. 2024]

Step 2: Slicing



Scaling and slicing: example on small vectors

$$A = \begin{bmatrix} 1.5625 & 8 & -3.6875 \end{bmatrix}, \qquad B = \begin{bmatrix} 1.3828125 \\ -7.625 \\ 3.625 \end{bmatrix}$$

Scale factors $\alpha = 2^4$ and $\beta = 2^3$.

Example set up: FP precision 8 bits, 4 slices, integer: 3 bits and a sign

$$\underbrace{\begin{bmatrix} 2^0 \cdot 1.1001000 \\ 2^3 \cdot 1.0000000 \\ -2^1 \cdot 1.1101100 \end{bmatrix}}_{A^T} \Rightarrow 2^4 \cdot \underbrace{\begin{bmatrix} \not{0}.\underline{000}\ \underline{110}\ \underline{010}\ \underline{000} \\ \not{0}.\underline{100}\ \underline{000}\ \underline{000}\ \underline{000} \\ -\not{0}.\underline{001}\ \underline{110}\ \underline{110}\ \underline{000} \end{bmatrix}}_{\text{Block fixed-point}} \Rightarrow 2^1 \cdot \underbrace{\begin{bmatrix} 000 \\ 100 \\ -001 \end{bmatrix}}_{A^T_{(1)}} + 2^{-2} \cdot \underbrace{\begin{bmatrix} 110 \\ 000 \\ -110 \end{bmatrix}}_{A^T_{(2)}} + 2^{-5} \cdot \underbrace{\begin{bmatrix} 010 \\ 000 \\ -110 \end{bmatrix}}_{A^T_{(3)}} + 2^{-8} \cdot \underbrace{\begin{bmatrix} 000 \\ 000 \\ 000 \end{bmatrix}}_{A^T_{(4)}}$$

$$\underbrace{\begin{bmatrix} 2^0 \cdot 1.0110001 \\ -2^2 \cdot 1.1110100 \\ 2^1 \cdot 1.1101000 \end{bmatrix}}_{B} \Rightarrow 2^3 \cdot \underbrace{\begin{bmatrix} \not{0}.\underline{001}\ \underline{011}\ \underline{000}\ \underline{100} \\ -\not{0}.\underline{111}\ \underline{101}\ \underline{000}\ \underline{000} \\ \not{0}.\underline{011}\ \underline{101}\ \underline{000}\ \underline{000} \end{bmatrix}}_{\text{Block fixed-point}} \Rightarrow 2^0 \cdot \underbrace{\begin{bmatrix} 001 \\ -111 \\ 011 \end{bmatrix}}_{B^{(1)}} + 2^{-3} \cdot \underbrace{\begin{bmatrix} 011 \\ -101 \\ 101 \end{bmatrix}}_{B^{(2)}} + 2^{-6} \cdot \underbrace{\begin{bmatrix} 000 \\ 000 \\ 000 \end{bmatrix}}_{B^{(3)}} + 2^{-9} \cdot \underbrace{\begin{bmatrix} 100 \\ 000 \\ 000 \end{bmatrix}}_{B^{(4)}}$$

FP64 $\to$ int8-int32 tensor core $\to$ FP64

We now have:

$$A \approx \mathrm{diag}(\alpha) \sum_{\ell=1}^{s_A} 2^{-\ell t} A_{(\ell)}, \qquad B \approx \sum_{h=1}^{s_B} 2^{-ht} B^{(h)} \,\mathrm{diag}(\beta).$$

Here:

- s_A and s_B are the number of slices of BFP A and B.
- $A_{(\ell)}$ is the ℓ-th slices of A.
- $B^{(h)}$ is the h-th slices of B.
- t is the precision of integers in $A_{(\ell)}$ and $B^{(h)}$.

FP64 $\rightarrow$ int8-int32 tensor core $\rightarrow$ FP64

Step 3: Integer products

$$C = AB \approx \left(\mathrm{diag}(\alpha)\sum_{\ell=1}^{s_A} 2^{-\ell t} A_{(\ell)}\right)\left(\sum_{h=1}^{s_B} 2^{-ht} B^{(h)}\,\mathrm{diag}(\beta)\right)$$

$$= \alpha\beta^T \circ \sum_{\ell=1}^{s_A}\sum_{h=1}^{s_B} 2^{-(\ell+h)t}\,\underline{A_{(\ell)}B^{(h)}}.$$

Step 4: Summation and rescaling (back in FP64)

$$C \approx \underline{\alpha\beta^T \circ \sum_{\ell=1}^{s_A}\sum_{h=1}^{s_B} 2^{-(\ell+h)t}} A_{(\ell)}B^{(h)}.$$

Example: products and their positions in BFP

$A_{(1)}B^{(1)}$	$2^{1}\ \cdot$	-00011111
$A_{(1)}B^{(2)}$	$2^{-2}\ \cdot$	-00011001
$A_{(2)}B^{(1)}$	$2^{-2}\ \cdot$	-00001100
$A_{(1)}B^{(3)}$	$2^{-5}\ \cdot$	00000000
$A_{(2)}B^{(2)}$	$2^{-5}\ \cdot$	-00001100
$A_{(3)}B^{(1)}$	$2^{-5}\ \cdot$	-00010000
$A_{(1)}B^{(4)}$	$2^{-8}\ \cdot$	00000000
$A_{(2)}B^{(3)}$	$2^{-8}\ \cdot$	00000000
$A_{(3)}B^{(2)}$	$2^{-8}\ \cdot$	-00011000
$A_{(4)}B^{(1)}$	$2^{-8}\ \cdot$	00000000
$A_{(2)}B^{(4)}$	$2^{-11}\ \cdot$	00011000
$A_{(3)}B^{(3)}$	$2^{-11}\ \cdot$	00000000
$A_{(4)}B^{(2)}$	$2^{-11}\ \cdot$	00000000
$A_{(3)}B^{(4)}$	$2^{-14}\ \cdot$	00001000
$A_{(4)}B^{(3)}$	$2^{-14}\ \cdot$	00000000
$A_{(4)}B^{(4)}$	$2^{-17}\ \cdot$	00000000
AB	$2^{-17}\ \cdot$	$-00100100000110100111000000$

Rounding error analysis 1: slicing

Error induced in slicing

Slicing causes the error in the virtual block floating-point precision, by taking a set number of slices s_A and s_B. The precision to the right of the binary point in BFP is $s_A t$ bits—**the more slices we take, the higher the precision of the input is**. However, BFP causes errors relative to the max value in a block.

For slicing A (analogous for B) we have the **absolute error**:

$$A = \Delta A + \operatorname{diag}(\alpha) \sum_{\ell=1}^{s_A} 2^{-\ell t} A_{(\ell)}, \qquad |\delta a_{ij}| < \alpha_i 2^{-s_A t}, \tag{1}$$

and the relative error (component wise)

$$\frac{|\delta a_{ij}|}{|a_{ij}|} < \frac{\alpha_i}{|a_{ij}|} 2^{-s_A t} \le \frac{2 \max_j |a_{ij}|}{\min_j |a_{ij}|} 2^{-s_A t} \le \kappa_A 2^{-s_A t}, \quad \kappa_A := 2 \max_i \frac{\max_j |a_{ij}|}{\min_j |a_{ij}|}$$

Rounding error analysis 2: product summation

Let $\widehat{C}$ be an FP approximation for $\alpha\beta^T \circ \sum_{\ell=1}^{s_A} \sum_{h=1}^{s_B} 2^{-(\ell+h)t} A_{(\ell)} B^{(h)}$.

We arrived at the bound for $\widehat{C}$:

$$|\widehat{C} - C| \lesssim \left(2^{-s_A t}\kappa_A + 2^{-s_B t}\kappa_B + (s_A s_B - 1)2^{-p}\right)|A||B|.$$

where p is the precision of the FP format used for accumulation (e.g. FP64).

Key take aways

- Large error if κ get large (large ranges of A and/or B).
- Taking more slices can help: inc. s_A or s_B. But more expensive.
- If $s_A t$ and $s_B t$ are large enough, 2^{-p} (accumulation error) dominates.

Experiment 1: 2-element vectors

As a minimal example, we consider the computation of the inner product $a^T b$, where

$$a = \begin{bmatrix} 2^{-\varphi} x \\ 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 2^{\varphi} y \\ 1 \end{bmatrix}, \qquad x, y \sim \mathcal{N}(0, 1).$$

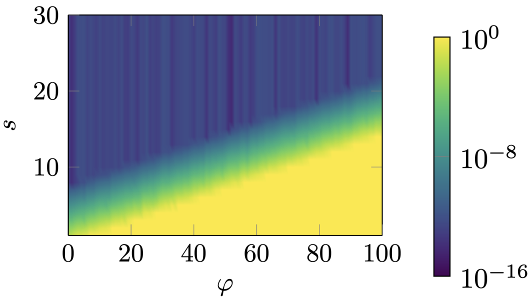
We plot:

$$\frac{|\widehat{c} - c|}{|c|},$$

where

- $\widehat{c}$ is computed with Ozaki scheme on 8-bit to 32-bit int tensor core.
- c is a reference solution computed using the MATLAB Symbolic Toolbox with 32 decimal digits of accuracy.

Experiment 1: 2-element vectors



The x-axis denotes the number of slices and the y-axis controls the wideness of the gap between the min and max exponents.

Key takeaways

- For $\varphi = 0$ about 7 slices are sufficient for FP64 accur.
- For $\varphi = 100$ about 20 slices are needed.

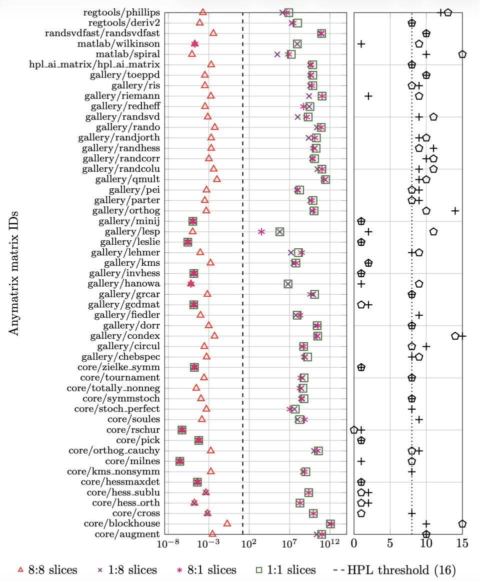
Experiment 2: Solving $Ax = b$ with block LU

- Take $A \in \mathbb{R}^{500 \times 500}$, one of the matrices from the `Anymatrix` collection [Higham and Mikaitis, 2021].
- $b \in \mathbb{R}^{500}$ with entries $[0, 1)$ from unif. distr.
- Use block LU with partial pivoting and block size 10.
- All mat-mul computed with the Ozaki scheme inside the LU iteration.

HPL error measure (used for TOP500)

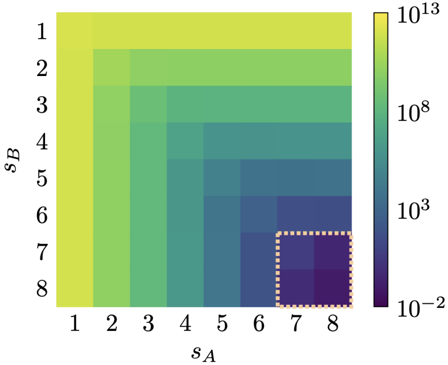
Report: $\frac{\|A\widehat{x}-b\|_\infty}{2u(\|A\|_\infty\|\widehat{x}\|_\infty+\|b\|_\infty)n}$ (HPL pass threshold < 16). Here $u = 2^{-53}$ for FP64.

Experiment 2: Solving $Ax = b$ with block LU



Experiment 2: Solving $Ax = b$ with block LU

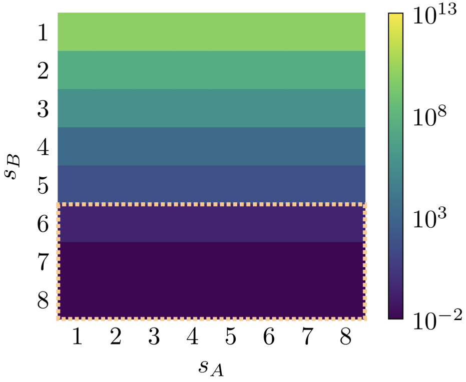
Error $\frac{\|A\widehat{x}-b\|_\infty}{2u(\|A\|_\infty\|\widehat{x}\|_\infty+\|b\|_\infty)n}$ in solving $Ax = b$ for A an Anymatrix matrix `core/blockhouse` with $n = 500$.



HPL threshold 16 passed for configs in dotted square.

Experiment 2: Solving $Ax = b$ with block LU

Error $\frac{\|A\widehat{x}-b\|_\infty}{2u(\|A\|_\infty\|\widehat{x}\|_\infty+\|b\|_\infty)n}$ in solving $Ax = b$ for A an Anymatrix matrix `core/cross` with $n = 500$.



HPL threshold 16 passed for configs in dotted square.

Summary

- Low-precision matrix multipliers are being used for general-purpose computation.
- We have
  - Built error analysis to check some intuitive predictions about the Ozaki scheme.
  - Experimentation to show where the scheme works or not.
  - Unbalanced slicing: future work.

Preprint

A. bdelfattah, J. Dongarra, M. Fasi, M. Mikaitis, and F. Tisseur. *Analysis of Floating-Point Matrix Multiplication Computed via Integer Arithmetic* . **Preprint, arXiv:2506.11277 [math.NA]**. Jun. 2025.

Slides at `http://mmikaitis.github.io/talks`

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SCA/HPCAsia 2026 Workshops, Osaka, Japan

(Extra slides) Experiment 3: matrices with increasing dyn. range

We take the matrices $A \in \mathbb{R}^{10\times k}$ and $B \in \mathbb{R}^{k\times 10}$ with

$$A_{ij} = a_{ij}e^{\varphi x_{ij}}, \quad B_{ij} = b_{ij}e^{\varphi y_{ij}}, \quad a_{ij}, b_{ij} \sim \mathcal{U}(-0.5, 0.5), \quad x_{ij}, y_{ij} \sim \mathcal{N}(0, 1),$$

The parameter φ allows us to control the exponent range of the elements in A and B.

We plot the max forw. rel. error (following [Ootomo, Ozaki, Yokota, 2024]):

$$\max_{i,j} \frac{|\widehat{c}_{ij} - c_{ij}|}{|c_{ij}|},$$

(Extra slides) Experiment 3: matrices with increasing dyn. range