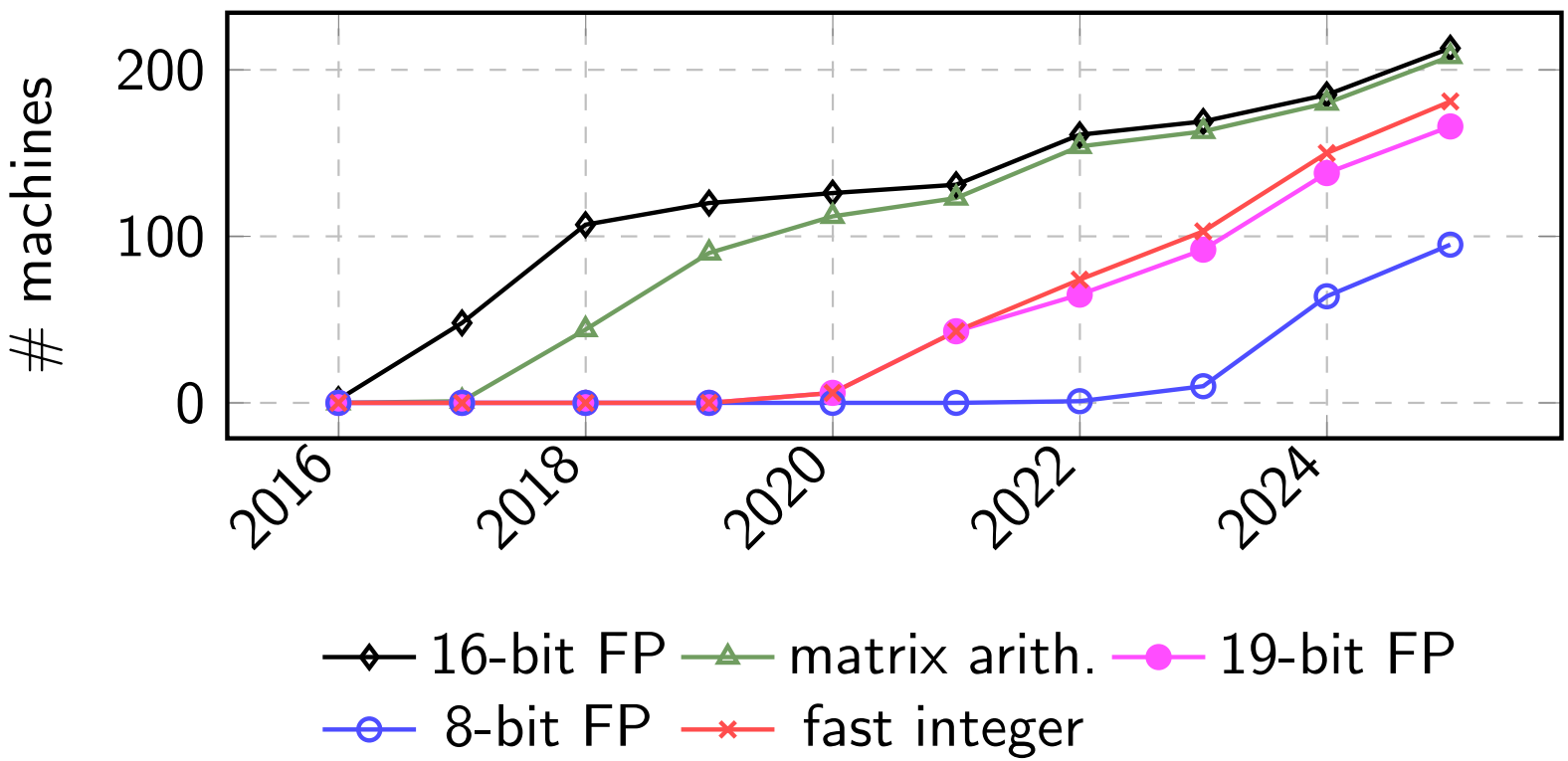
Error Analysis of Matrix Multiplication with Narrow Range Floating-Point Arithmetic

Mantas Mikaitis

School of Computer Science, University of Leeds, Leeds, UK
Joint work with Theo Mary, CNRS, Paris, France

The 30th Biennial Numerical Analysis Conference
Glasgow, UK
26 June, 2025

8-bit floating point on the TOP500 (June 2025)



Devices counted: P100, V100, A100, H100, MI210, MI250X, MI300X, Intel Data Center GPU, from `https://www.top500.org`.

NVIDIA Blackwell throughputs (FLOPS)

fp8 (9×10^{15}) fp16 (4.5×10^{15}) fp64 (0.04×10^{15}).

IEEE P3109: 8-bit format on one slide

C.4 Value Table: P4, emin $= -7$, emax $= 7$

0x00 = 0_0000_000 = 0.0	0x40 = 0_1000_000 = +0b1.000×2^0 = 1.0	0x80 = 1_0000_000 = NaN	0xc0 = 1_1000_000 = −0b1.000×2^0 = -1.0
0x01 = 0_0000_001 = +0b0.001×2^-7 = 0.0009765625	0x41 = 0_1000_001 = +0b1.001×2^0 = 1.125	0x81 = 1_0000_001 = −0b0.001×2^-7 = -0.0009765625	0xc1 = 1_1000_001 = −0b1.001×2^0 = -1.125
0x02 = 0_0000_010 = +0b0.010×2^-7 = 0.001953125	0x42 = 0_1000_010 = +0b1.010×2^0 = 1.25	0x82 = 1_0000_010 = −0b0.010×2^-7 = -0.001953125	0xc2 = 1_1000_010 = −0b1.010×2^0 = -1.25
0x03 = 0_0000_011 = +0b0.011×2^-7 = 0.0029296875	0x43 = 0_1000_011 = +0b1.011×2^0 = 1.375	0x83 = 1_0000_011 = −0b0.011×2^-7 = -0.0029296875	0xc3 = 1_1000_011 = −0b1.011×2^0 = -1.375
0x04 = 0_0000_100 = +0b0.100×2^-7 = 0.00390625	0x44 = 0_1000_100 = +0b1.100×2^0 = 1.5	0x84 = 1_0000_100 = −0b0.100×2^-7 = -0.00390625	0xc4 = 1_1000_100 = −0b1.100×2^0 = -1.5
0x05 = 0_0000_101 = +0b0.101×2^-7 = 0.0048828125	0x45 = 0_1000_101 = +0b1.101×2^0 = 1.625	0x85 = 1_0000_101 = −0b0.101×2^-7 = -0.0048828125	0xc5 = 1_1000_101 = −0b1.101×2^0 = -1.625
0x06 = 0_0000_110 = +0b0.110×2^-7 = 0.005859375	0x46 = 0_1000_110 = +0b1.110×2^0 = 1.75	0x86 = 1_0000_110 = −0b0.110×2^-7 = -0.005859375	0xc6 = 1_1000_110 = −0b1.110×2^0 = -1.75
0x07 = 0_0000_111 = +0b0.111×2^-7 = 0.0068359375	0x47 = 0_1000_111 = +0b1.111×2^0 = 1.875	0x87 = 1_0000_111 = −0b0.111×2^-7 = -0.0068359375	0xc7 = 1_1000_111 = −0b1.111×2^0 = -1.875
0x08 = 0_0001_000 = +0b1.000×2^-7 = 0.0078125	0x48 = 0_1001_000 = +0b1.000×2^1 = 2.0	0x88 = 1_0001_000 = −0b1.000×2^-7 = -0.0078125	0xc8 = 1_1001_000 = −0b1.000×2^1 = -2.0
0x09 = 0_0001_001 = +0b1.001×2^-7 = 0.0087890625	0x49 = 0_1001_001 = +0b1.001×2^1 = 2.25	0x89 = 1_0001_001 = −0b1.001×2^-7 = -0.0087890625	0xc9 = 1_1001_001 = −0b1.001×2^1 = -2.25
0x0a = 0_0001_010 = +0b1.010×2^-7 = 0.009765625	0x4a = 0_1001_010 = +0b1.010×2^1 = 2.5	0x8a = 1_0001_010 = −0b1.010×2^-7 = -0.009765625	0xca = 1_1001_010 = −0b1.010×2^1 = -2.5
0x0b = 0_0001_011 = +0b1.011×2^-7 = 0.0107421875	0x4b = 0_1001_011 = +0b1.011×2^1 = 2.75	0x8b = 1_0001_011 = −0b1.011×2^-7 = -0.0107421875	0xcb = 1_1001_011 = −0b1.011×2^1 = -2.75
0x0c = 0_0001_100 = +0b1.100×2^-7 = 0.01171875	0x4c = 0_1001_100 = +0b1.100×2^1 = 3.0	0x8c = 1_0001_100 = −0b1.100×2^-7 = -0.01171875	0xcc = 1_1001_100 = −0b1.100×2^1 = -3.0
0x0d = 0_0001_101 = +0b1.101×2^-7 = 0.0126953125	0x4d = 0_1001_101 = +0b1.101×2^1 = 3.25	0x8d = 1_0001_101 = −0b1.101×2^-7 = -0.0126953125	0xcd = 1_1001_101 = −0b1.101×2^1 = -3.25
0x0e = 0_0001_110 = +0b1.110×2^-7 = 0.013671875	0x4e = 0_1001_110 = +0b1.110×2^1 = 3.5	0x8e = 1_0001_110 = −0b1.110×2^-7 = -0.013671875	0xce = 1_1001_110 = −0b1.110×2^1 = -3.5
0x0f = 0_0001_111 = +0b1.111×2^-7 = 0.0146484375	0x4f = 0_1001_111 = +0b1.111×2^1 = 3.75	0x8f = 1_0001_111 = −0b1.111×2^-7 = -0.0146484375	0xcf = 1_1001_111 = −0b1.111×2^1 = -3.75
0x10 = 0_0010_000 = +0b1.000×2^-6 = 0.015625	0x50 = 0_1010_000 = +0b1.000×2^2 = 4.0	0x90 = 1_0010_000 = −0b1.000×2^-6 = -0.015625	0xd0 = 1_1010_000 = −0b1.000×2^2 = -4.0
0x11 = 0_0010_001 = +0b1.001×2^-6 = 0.017578125	0x51 = 0_1010_001 = +0b1.001×2^2 = 4.5	0x91 = 1_0010_001 = −0b1.001×2^-6 = -0.017578125	0xd1 = 1_1010_001 = −0b1.001×2^2 = -4.5
0x12 = 0_0010_010 = +0b1.010×2^-6 = 0.01953125	0x52 = 0_1010_010 = +0b1.010×2^2 = 5.0	0x92 = 1_0010_010 = −0b1.010×2^-6 = -0.01953125	0xd2 = 1_1010_010 = −0b1.010×2^2 = -5.0
0x13 = 0_0010_011 = +0b1.011×2^-6 = 0.021484375	0x53 = 0_1010_011 = +0b1.011×2^2 = 5.5	0x93 = 1_0010_011 = −0b1.011×2^-6 = -0.021484375	0xd3 = 1_1010_011 = −0b1.011×2^2 = -5.5
0x14 = 0_0010_100 = +0b1.100×2^-6 = 0.0234375	0x54 = 0_1010_100 = +0b1.100×2^2 = 6.0	0x94 = 1_0010_100 = −0b1.100×2^-6 = -0.0234375	0xd4 = 1_1010_100 = −0b1.100×2^2 = -6.0
0x15 = 0_0010_101 = +0b1.101×2^-6 = 0.025390625	0x55 = 0_1010_101 = +0b1.101×2^2 = 6.5	0x95 = 1_0010_101 = −0b1.101×2^-6 = -0.025390625	0xd5 = 1_1010_101 = −0b1.101×2^2 = -6.5
0x16 = 0_0010_110 = +0b1.110×2^-6 = 0.02734375	0x56 = 0_1010_110 = +0b1.110×2^2 = 7.0	0x96 = 1_0010_110 = −0b1.110×2^-6 = -0.02734375	0xd6 = 1_1010_110 = −0b1.110×2^2 = -7.0
0x17 = 0_0010_111 = +0b1.111×2^-6 = 0.029296875	0x57 = 0_1010_111 = +0b1.111×2^2 = 7.5	0x97 = 1_0010_111 = −0b1.111×2^-6 = -0.029296875	0xd7 = 1_1010_111 = −0b1.111×2^2 = -7.5
0x18 = 0_0011_000 = +0b1.000×2^-5 = 0.03125	0x58 = 0_1011_000 = +0b1.000×2^3 = 8.0	0x98 = 1_0011_000 = −0b1.000×2^-5 = -0.03125	0xd8 = 1_1011_000 = −0b1.000×2^3 = -8.0
0x19 = 0_0011_001 = +0b1.001×2^-5 = 0.03515625	0x59 = 0_1011_001 = +0b1.001×2^3 = 9.0	0x99 = 1_0011_001 = −0b1.001×2^-5 = -0.03515625	0xd9 = 1_1011_001 = −0b1.001×2^3 = -9.0
0x1a = 0_0011_010 = +0b1.010×2^-5 = 0.0390625	0x5a = 0_1011_010 = +0b1.010×2^3 = 10.0	0x9a = 1_0011_010 = −0b1.010×2^-5 = -0.0390625	0xda = 1_1011_010 = −0b1.010×2^3 = -10.0
0x1b = 0_0011_011 = +0b1.011×2^-5 = 0.04296875	0x5b = 0_1011_011 = +0b1.011×2^3 = 11.0	0x9b = 1_0011_011 = −0b1.011×2^-5 = -0.04296875	0xdb = 1_1011_011 = −0b1.011×2^3 = -11.0
0x1c = 0_0011_100 = +0b1.100×2^-5 = 0.046875	0x5c = 0_1011_100 = +0b1.100×2^3 = 12.0	0x9c = 1_0011_100 = −0b1.100×2^-5 = -0.046875	0xdc = 1_1011_100 = −0b1.100×2^3 = -12.0
0x1d = 0_0011_101 = +0b1.101×2^-5 = 0.05078125	0x5d = 0_1011_101 = +0b1.101×2^3 = 13.0	0x9d = 1_0011_101 = −0b1.101×2^-5 = -0.05078125	0xdd = 1_1011_101 = −0b1.101×2^3 = -13.0
0x1e = 0_0011_110 = +0b1.110×2^-5 = 0.0546875	0x5e = 0_1011_110 = +0b1.110×2^3 = 14.0	0x9e = 1_0011_110 = −0b1.110×2^-5 = -0.0546875	0xde = 1_1011_110 = −0b1.110×2^3 = -14.0
0x1f = 0_0011_111 = +0b1.111×2^-5 = 0.05859375	0x5f = 0_1011_111 = +0b1.111×2^3 = 15.0	0x9f = 1_0011_111 = −0b1.111×2^-5 = -0.05859375	0xdf = 1_1011_111 = −0b1.111×2^3 = -15.0
0x20 = 0_0100_000 = +0b1.000×2^-4 = 0.0625	0x60 = 0_1100_000 = +0b1.000×2^4 = 16.0	0xa0 = 1_0100_000 = −0b1.000×2^-4 = -0.0625	0xe0 = 1_1100_000 = −0b1.000×2^4 = -16.0
0x21 = 0_0100_001 = +0b1.001×2^-4 = 0.0703125	0x61 = 0_1100_001 = +0b1.001×2^4 = 18.0	0xa1 = 1_0100_001 = −0b1.001×2^-4 = -0.0703125	0xe1 = 1_1100_001 = −0b1.001×2^4 = -18.0
0x22 = 0_0100_010 = +0b1.010×2^-4 = 0.078125	0x62 = 0_1100_010 = +0b1.010×2^4 = 20.0	0xa2 = 1_0100_010 = −0b1.010×2^-4 = -0.078125	0xe2 = 1_1100_010 = −0b1.010×2^4 = -20.0
0x23 = 0_0100_011 = +0b1.011×2^-4 = 0.0859375	0x63 = 0_1100_011 = +0b1.011×2^4 = 22.0	0xa3 = 1_0100_011 = −0b1.011×2^-4 = -0.0859375	0xe3 = 1_1100_011 = −0b1.011×2^4 = -22.0
0x24 = 0_0100_100 = +0b1.100×2^-4 = 0.09375	0x64 = 0_1100_100 = +0b1.100×2^4 = 24.0	0xa4 = 1_0100_100 = −0b1.100×2^-4 = -0.09375	0xe4 = 1_1100_100 = −0b1.100×2^4 = -24.0
0x25 = 0_0100_101 = +0b1.101×2^-4 = 0.1015625	0x65 = 0_1100_101 = +0b1.101×2^4 = 26.0	0xa5 = 1_0100_101 = −0b1.101×2^-4 = -0.1015625	0xe5 = 1_1100_101 = −0b1.101×2^4 = -26.0
0x26 = 0_0100_110 = +0b1.110×2^-4 = 0.109375	0x66 = 0_1100_110 = +0b1.110×2^4 = 28.0	0xa6 = 1_0100_110 = −0b1.110×2^-4 = -0.109375	0xe6 = 1_1100_110 = −0b1.110×2^4 = -28.0
0x27 = 0_0100_111 = +0b1.111×2^-4 = 0.1171875	0x67 = 0_1100_111 = +0b1.111×2^4 = 30.0	0xa7 = 1_0100_111 = −0b1.111×2^-4 = -0.1171875	0xe7 = 1_1100_111 = −0b1.111×2^4 = -30.0
0x28 = 0_0101_000 = +0b1.000×2^-3 = 0.125	0x68 = 0_1101_000 = +0b1.000×2^5 = 32.0	0xa8 = 1_0101_000 = −0b1.000×2^-3 = -0.125	0xe8 = 1_1101_000 = −0b1.000×2^5 = -32.0
0x29 = 0_0101_001 = +0b1.001×2^-3 = 0.140625	0x69 = 0_1101_001 = +0b1.001×2^5 = 36.0	0xa9 = 1_0101_001 = −0b1.001×2^-3 = -0.140625	0xe9 = 1_1101_001 = −0b1.001×2^5 = -36.0
0x2a = 0_0101_010 = +0b1.010×2^-3 = 0.15625	0x6a = 0_1101_010 = +0b1.010×2^5 = 40.0	0xaa = 1_0101_010 = −0b1.010×2^-3 = -0.15625	0xea = 1_1101_010 = −0b1.010×2^5 = -40.0
0x2b = 0_0101_011 = +0b1.011×2^-3 = 0.171875	0x6b = 0_1101_011 = +0b1.011×2^5 = 44.0	0xab = 1_0101_011 = −0b1.011×2^-3 = -0.171875	0xeb = 1_1101_011 = −0b1.011×2^5 = -44.0
0x2c = 0_0101_100 = +0b1.100×2^-3 = 0.1875	0x6c = 0_1101_100 = +0b1.100×2^5 = 48.0	0xac = 1_0101_100 = −0b1.100×2^-3 = -0.1875	0xec = 1_1101_100 = −0b1.100×2^5 = -48.0
0x2d = 0_0101_101 = +0b1.101×2^-3 = 0.203125	0x6d = 0_1101_101 = +0b1.101×2^5 = 52.0	0xad = 1_0101_101 = −0b1.101×2^-3 = -0.203125	0xed = 1_1101_101 = −0b1.101×2^5 = -52.0
0x2e = 0_0101_110 = +0b1.110×2^-3 = 0.21875	0x6e = 0_1101_110 = +0b1.110×2^5 = 56.0	0xae = 1_0101_110 = −0b1.110×2^-3 = -0.21875	0xee = 1_1101_110 = −0b1.110×2^5 = -56.0
0x2f = 0_0101_111 = +0b1.111×2^-3 = 0.234375	0x6f = 0_1101_111 = +0b1.111×2^5 = 60.0	0xaf = 1_0101_111 = −0b1.111×2^-3 = -0.234375	0xef = 1_1101_111 = −0b1.111×2^5 = -60.0
0x30 = 0_0110_000 = +0b1.000×2^-2 = 0.25	0x70 = 0_1110_000 = +0b1.000×2^6 = 64.0	0xb0 = 1_0110_000 = −0b1.000×2^-2 = -0.25	0xf0 = 1_1110_000 = −0b1.000×2^6 = -64.0
0x31 = 0_0110_001 = +0b1.001×2^-2 = 0.28125	0x71 = 0_1110_001 = +0b1.001×2^6 = 72.0	0xb1 = 1_0110_001 = −0b1.001×2^-2 = -0.28125	0xf1 = 1_1110_001 = −0b1.001×2^6 = -72.0
0x32 = 0_0110_010 = +0b1.010×2^-2 = 0.3125	0x72 = 0_1110_010 = +0b1.010×2^6 = 80.0	0xb2 = 1_0110_010 = −0b1.010×2^-2 = -0.3125	0xf2 = 1_1110_010 = −0b1.010×2^6 = -80.0
0x33 = 0_0110_011 = +0b1.011×2^-2 = 0.34375	0x73 = 0_1110_011 = +0b1.011×2^6 = 88.0	0xb3 = 1_0110_011 = −0b1.011×2^-2 = -0.34375	0xf3 = 1_1110_011 = −0b1.011×2^6 = -88.0
0x34 = 0_0110_100 = +0b1.100×2^-2 = 0.375	0x74 = 0_1110_100 = +0b1.100×2^6 = 96.0	0xb4 = 1_0110_100 = −0b1.100×2^-2 = -0.375	0xf4 = 1_1110_100 = −0b1.100×2^6 = -96.0
0x35 = 0_0110_101 = +0b1.101×2^-2 = 0.40625	0x75 = 0_1110_101 = +0b1.101×2^6 = 104.0	0xb5 = 1_0110_101 = −0b1.101×2^-2 = -0.40625	0xf5 = 1_1110_101 = −0b1.101×2^6 = -104.0
0x36 = 0_0110_110 = +0b1.110×2^-2 = 0.4375	0x76 = 0_1110_110 = +0b1.110×2^6 = 112.0	0xb6 = 1_0110_110 = −0b1.110×2^-2 = -0.4375	0xf6 = 1_1110_110 = −0b1.110×2^6 = -112.0
0x37 = 0_0110_111 = +0b1.111×2^-2 = 0.46875	0x77 = 0_1110_111 = +0b1.111×2^6 = 120.0	0xb7 = 1_0110_111 = −0b1.111×2^-2 = -0.46875	0xf7 = 1_1110_111 = −0b1.111×2^6 = -120.0
0x38 = 0_0111_000 = +0b1.000×2^-1 = 0.5	0x78 = 0_1111_000 = +0b1.000×2^7 = 128.0	0xb8 = 1_0111_000 = −0b1.000×2^-1 = -0.5	0xf8 = 1_1111_000 = −0b1.000×2^7 = -128.0
0x39 = 0_0111_001 = +0b1.001×2^-1 = 0.5625	0x79 = 0_1111_001 = +0b1.001×2^7 = 144.0	0xb9 = 1_0111_001 = −0b1.001×2^-1 = -0.5625	0xf9 = 1_1111_001 = −0b1.001×2^7 = -144.0
0x3a = 0_0111_010 = +0b1.010×2^-1 = 0.625	0x7a = 0_1111_010 = +0b1.010×2^7 = 160.0	0xba = 1_0111_010 = −0b1.010×2^-1 = -0.625	0xfa = 1_1111_010 = −0b1.010×2^7 = -160.0
0x3b = 0_0111_011 = +0b1.011×2^-1 = 0.6875	0x7b = 0_1111_011 = +0b1.011×2^7 = 176.0	0xbb = 1_0111_011 = −0b1.011×2^-1 = -0.6875	0xfb = 1_1111_011 = −0b1.011×2^7 = -176.0
0x3c = 0_0111_100 = +0b1.100×2^-1 = 0.75	0x7c = 0_1111_100 = +0b1.100×2^7 = 192.0	0xbc = 1_0111_100 = −0b1.100×2^-1 = -0.75	0xfc = 1_1111_100 = −0b1.100×2^7 = -192.0
0x3d = 0_0111_101 = +0b1.101×2^-1 = 0.8125	0x7d = 0_1111_101 = +0b1.101×2^7 = 208.0	0xbd = 1_0111_101 = −0b1.101×2^-1 = -0.8125	0xfd = 1_1111_101 = −0b1.101×2^7 = -208.0
0x3e = 0_0111_110 = +0b1.110×2^-1 = 0.875	0x7e = 0_1111_110 = +0b1.110×2^7 = 224.0	0xbe = 1_0111_110 = −0b1.110×2^-1 = -0.875	0xfe = 1_1111_110 = −0b1.110×2^7 = -224.0
0x3f = 0_0111_111 = +0b1.111×2^-1 = 0.9375	0x7f = 0_1111_111 = +Inf	0xbf = 1_0111_111 = −0b1.111×2^-1 = -0.9375	0xff = 1_1111_111 = −Inf

4/6/8/16-bit floating point formats have narrow ranges

Format	precision	min pos.	max pos.	u
binary64 (double)	53	2^{-1022}	$\sim 1.798 \times 10^{308}$	2^{-53}
binary32 (single)	24	2^{-126}	$\sim 3.403 \times 10^{38}$	2^{-24}
tf32 (19-bit)	11	2^{-126}	$\sim 3.401 \times 10^{38}$	2^{-11}
bfloat16	8	2^{-126}	$\sim 3.389 \times 10^{38}$	2^{-8}
binary16	11	2^{-14}	65504	2^{-11}
fp8-E4M3	4	2^{-6}	448	2^{-4}
fp8-E5M2	3	2^{-14}	57344	2^{-3}
fp6-E2M3	4	2^{0}	7.5	2^{-4}
fp6-E3M2	3	2^{-2}	28	2^{-3}
fp4-E2M1	2	2^{0}	6	2^{-2}

Mixed-precision matrix multipliers

Formats with narrow ranges are available in matrix multiply operation.

$$\underbrace{D}_{\substack{\text{binary16 or}\\ \text{binary32}}} = \underbrace{C}_{\substack{\text{binary16 or}\\ \text{binary32}}} + \underbrace{A}_{\text{8-bit FP}} \times \underbrace{B}_{\text{8-bit FP}},$$

$$\underbrace{\begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix}}_{\substack{\text{binary16 or}\\ \text{binary32}}} = \underbrace{\begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix}}_{\substack{\text{binary16 or}\\ \text{binary32}}} + \underbrace{\begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix}}_{\text{8-bit FP}} \times \underbrace{\begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix}}_{\text{8-bit FP}}$$

Hardware matrix multipliers in mixed precision

- Example above is 4×4, but dimensions differ across architectures.
- Internal dot product precision, rounding, and subnormal support.

While the input formats have narrow ranges, the output is less constrained on both the precision and range.

Part 1: Basic single-word algorithm

Single-word algorithm

Goal: Given A and B, matrices in, for example, binary64, multiply them accurately using mixed-precision MMAs.

1. Scale input matrices A and B.
2. Round input matrices to the *input format*.
3. Multiply scaled and rounded A and B in the *accumulation format*.
4. Scale the output matrix.

$$C = \Lambda^{-1}\Big(\mathrm{fl}(\Lambda A)\mathrm{fl}(BM)\Big)M^{-1}$$

- Λ and M are nonsingular diagonal matrices with diagonal coefficients λ_i and μ_i respectively.
- Scale coefficients λ_i and μ_i are powers of two.

Single-word algorithm

Let θ be the maximum value we can afford in the scaled A and B.

Scaling by powers of two means the maximum entry per row of A or column of B is in $(\theta/2, \theta]$.

We should maximise θ to reduce number of underflows, but at the same time remove possibility of overflow.

Choose:

$$\theta = \min(f_{\max}, \sqrt{F_{\max}/n}).$$

which avoids overflow in the input and in the accumulation of n products.

Single-word algorithm: an example

- Take $A \in \mathbb{R}^{4\times 4}$ and $B \in \mathbb{R}^{4\times 4}$.
- Set fp8-E4M3 as the *input format* with $f_{\max} = 448$.
- Set binary16 as the *accumulation format* with $F_{\max} = 65504$.
- No subnormal floating-point numbers.
- This gives $\min(448, \sqrt{65504/4}) = \min(448, 127.9687) \approx 127 = \theta$.

Scaling factors

In this case before rounding matrices to the *input format* we need to scale them such that 127 is the maximum value that appears.

- 127 is lower than $f_{\max} = 448$ - no *input format* overflows.
- $127 \times 127 = 16129$ and if we accumulate four such products we get $64616 < F_{\max} = 65504$. No *accumulation format* overflows.

Single-word algorithm: an example

Take

$$A = \begin{bmatrix} 500 & 1 & 1 & 2^{-6} \\ 128 & 128 & 128 & 128 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 128 & 1 & 1 \\ 1 & 128 & 1 & 1 \\ 1 & 128 & 1 & 1 \\ 1 & 128 & 1 & 1 \end{bmatrix}.$$

We have

$$AB = \begin{bmatrix} 502.015625 & 64258 & 502.015625 & 502.015625 \\ 512 & 65536 & 512 & 512 \\ 4 & 512 & 4 & 4 \\ 4 & 512 & 4 & 4 \end{bmatrix}.$$

Overflows in the above example if no scaling is applied

(Input) $500 > f_{\max} = 448$ and (output) $65536 > F_{\max} = 65504$.

Single-word algorithm: an example

$$C = \Lambda^{-1}\Big(\mathrm{fl}(\Lambda A)\mathrm{fl}(BM)\Big)M^{-1}, \quad \theta = 127$$

Step 1: Scale A and B.

$$\Lambda A = \begin{bmatrix} 2^{-2} & 0 & 0 & 0 \\ 0 & 2^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 500 & 1 & 1 & 2^{-6} \\ 128 & 128 & 128 & 128 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 125 & 2^{-2} & 2^{-2} & 2^{-8} \\ 64 & 64 & 64 & 64 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$BM = \begin{bmatrix} 1 & 128 & 1 & 1 \\ 1 & 128 & 1 & 1 \\ 1 & 128 & 1 & 1 \\ 1 & 128 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 64 & 1 & 1 \\ 1 & 64 & 1 & 1 \\ 1 & 64 & 1 & 1 \\ 1 & 64 & 1 & 1 \end{bmatrix}$$

How the scale coefficients are calculated

For example, take the first row of A. The largest value is 500 and we need to get it below $\theta = 127$. $\lambda_1 = 2^{\lfloor \log_2(127/500) \rfloor} = 2^{-2}$.

Single-word algorithm: an example

$$C = \Lambda^{-1}\Big(\mathrm{fl}(\Lambda A)\mathrm{fl}(BM)\Big)M^{-1}$$

Step 2: Round to the *input format* fp8-E4M3 ($f_{\min} = 2^{-6}$).

$$\mathrm{fl}(\Lambda A) = \mathrm{fl}\left(\begin{bmatrix} 125 & 2^{-2} & 2^{-2} & 2^{-8} \\ 64 & 64 & 64 & 64 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 125 & 2^{-2} & 2^{-2} & \mathbf{0} \\ 64 & 64 & 64 & 64 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathrm{fl}(BM) = \mathrm{fl}\left(\begin{bmatrix} 1 & 64 & 1 & 1 \\ 1 & 64 & 1 & 1 \\ 1 & 64 & 1 & 1 \\ 1 & 64 & 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 64 & 1 & 1 \\ 1 & 64 & 1 & 1 \\ 1 & 64 & 1 & 1 \\ 1 & 64 & 1 & 1 \end{bmatrix}$$

Underflow in the above example

Notice that since subnormals are off, numbers $\leq f_{\min}/2$ will round to zero, causing underflow. This happened to $\Lambda A(1,4) = 2^{-8}$, which resulted from scaling the first row of A, where originally $A(1,4) = 2^{-6}$.

Single-word algorithm: an example

$$C = \Lambda^{-1}\Big(\mathrm{fl}(\Lambda A)\mathrm{fl}(BM)\Big)M^{-1}$$

Step 3: Perform matrix multiply in the *accumulation format* binary16 ($T = 11$, $F_{\max} = 65504$).

$$\begin{bmatrix} 125 & 2^{-2} & 2^{-2} & 0 \\ 64 & 64 & 64 & 64 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 64 & 1 & 1 \\ 1 & 64 & 1 & 1 \\ 1 & 64 & 1 & 1 \\ 1 & 64 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 125.5 & 8032 & 125.5 & 125.5 \\ 256 & 16384 & 256 & 256 \\ 4 & 256 & 4 & 4 \\ 4 & 256 & 4 & 4 \end{bmatrix}$$

Single-word algorithm: an example

$$C = \Lambda^{-1}\Big(\mathrm{fl}(\Lambda A)\mathrm{fl}(BM)\Big)M^{-1}$$

Step 4: Undo the scaling.

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 125.5 & 8032 & 125.5 & 125.5 \\ 256 & 16384 & 256 & 256 \\ 4 & 256 & 4 & 4 \\ 4 & 256 & 4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 502 & 64256 & 502 & 502 \\ 512 & 65536 & 512 & 512 \\ 4 & 512 & 4 & 4 \\ 4 & 512 & 4 & 4 \end{bmatrix}$$

Single-word algorithm: an example

$$C = \Lambda^{-1}\Big(\mathrm{fl}(\Lambda A)\mathrm{fl}(BM)\Big)M^{-1}$$

Comparison. Our result computed with mixed-precision MMA:

$$AB \approx \begin{bmatrix} \mathbf{502} & \mathbf{64256} & \mathbf{502} & \mathbf{502} \\ 512 & 65536 & 512 & 512 \\ 4 & 512 & 4 & 4 \\ 4 & 512 & 4 & 4 \end{bmatrix}$$

And the exact result

$$AB = \begin{bmatrix} \mathbf{502.015625} & \mathbf{64258} & \mathbf{502.015625} & \mathbf{502.015625} \\ 512 & 65536 & 512 & 512 \\ 4 & 512 & 4 & 4 \\ 4 & 512 & 4 & 4 \end{bmatrix}$$

Part 2: Error analysis

Matrix Multiply-Accumulate (MMA)

Model 1

The following model describes a mixed-precision MMA operation to compute $C = AB$, assuming round-to-nearest ties-to-even is used. We have two FP formats:

- *Input format* with precision t, unit roundoff $u = 2^{-t}$, exponent in $[e_{min}, e_{max}]$, range of normalized values $\pm[f_{\min}, f_{\max}]$. The maximum distance between any number in $[-f_{\min}, f_{\min}]$ and the nearest FP number is

$$g_{\min} = \begin{cases} f_{\min}/2 & \text{if subnormals are not available} \\ uf_{\min} & \text{with subnormals (gradual underflow)} \end{cases}$$

- *Accumulation format* with $T \geq t$, $U = 2^{-T}$, exponent in $[E_{\min}, E_{\max}] \supseteq [e_{\min}, e_{\max}]$, and range of norm. numbers $\pm[F_{\min}, F_{\max}]$. The maximum distance between any number in $[-F_{\min}, F_{\min}]$ and the nearest FP number is

$$G_{\min} = \begin{cases} F_{\min}/2 & \text{if subnormals are not available} \\ UF_{\min} & \text{with subnormals (gradual underflow)} \end{cases}$$

Models of worst-case rounding errors

Rounding error model based on [Demmel, 1984]

Take $x, y \in \mathbb{R}$. Assuming no overflows occur, the rounding operator to the *input format* is described as

$$\mathsf{fl}(x) = x(1+\delta) + \eta, \quad |\delta| \leq u, \quad |\eta| \leq g_{\min}, \quad \eta\delta = 0,$$

and arithmetic operations in the *accumulation format* as

$$\mathsf{FL}(x \,\mathsf{op}\, y) = (x \,\mathsf{op}\, y)(1+\delta) + \eta, \quad |\delta| \leq U, \quad |\eta| \leq G_{\min}, \quad \eta\delta = 0,$$

with $\mathsf{op} \in \{+, -, \times, \div\}$.

Here $\eta\delta = 0$ accounts for only one type of error, rounding or overflow.

Norm-wise error bound

Single-word algorithm:

$$\|\widehat{C} - AB\|_\infty \lesssim \left(2u + nU + 4n^2\theta^{-1}g_{\min} + 4n^2\theta^{-2}G_{\min}\right)\|A\|_\infty\|B\|_\infty.$$

Part 3: Multi-word algorithm

Double-word algorithm: an example

Step 2: Round to the *input format*, in **double-word representation**.

We will round each ΛA and BM to two fp8-E4M3 matrices instead of one.

Compute the first word (first of the two matrices):

$$A^{(0)} = \mathrm{fl}(\Lambda A) = \mathrm{fl}\left(\begin{bmatrix} 125 & 2^{-2} & 2^{-2} & 2^{-8} \\ 64 & 64 & 64 & 64 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 125 & 2^{-2} & 2^{-2} & \mathbf{0} \\ 64 & 64 & 64 & 64 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$B^{(0)} = \mathrm{fl}(BM) = \mathrm{fl}\left(\begin{bmatrix} 1 & 64 & 1 & 1 \\ 1 & 64 & 1 & 1 \\ 1 & 64 & 1 & 1 \\ 1 & 64 & 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 64 & 1 & 1 \\ 1 & 64 & 1 & 1 \\ 1 & 64 & 1 & 1 \\ 1 & 64 & 1 & 1 \end{bmatrix}$$

Double-word algorithm: an example

Step 2: Round to the *input format* fp8-E4M3, in **double-word representation**.

Compute the second word (rounding/underflow error in the first step):

$$A^{(1)} = \mathrm{fl}((\Lambda A - A^{(0)})/u^1) =$$

$$\mathrm{fl}\left(\left(\begin{bmatrix} 125 & 2^{-2} & 2^{-2} & 2^{-8} \\ 64 & 64 & 64 & 64 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 125 & 2^{-2} & 2^{-2} & \mathbf{0} \\ 64 & 64 & 64 & 64 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}\right) ./2^{-4}\right) = \begin{bmatrix} 0 & 0 & 0 & \mathbf{2^{-4}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since $B^{(0)} = BM, B^{(1)} = \mathrm{zeros}(4,4)$.

Extra scaling

Notice the division by $u^1 = 2^{-4}$ before rounding, which is done to reduce underflows in the input format. In general, the multi-word split is $A^{(i)} = \mathrm{fl}\left(\left(\Lambda A - \sum_{k=0}^{i-1} u^k A^{(k)}\right)/u^i\right)$.

Double-word algorithm: an example

Step 3: Perform matrix products and add them in the *accumulation format* binary16.

p-word case

After splitting ΛA and BM into $A^{(0)}, \ldots, A^{(p-1)}$ and $B^{(0)}, \ldots, B^{(p-1)}$, approximate matrix multiply by $p(p+1)/2$ products

$$C \approx \Lambda^{-1} \left(\sum_{i+j<p} u^{i+j} A^{(i)} B^{(j)} \right) M^{-1}.$$

In our double-word case

$$A^{(0)}B^{(0)} + uA^{(1)}B^{(0)} =$$

$$\begin{bmatrix} 125 & 2^{-2} & 2^{-2} & 0 \\ 64 & 64 & 64 & 64 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 64 & 1 & 1 \\ 1 & 64 & 1 & 1 \\ 1 & 64 & 1 & 1 \\ 1 & 64 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & \mathbf{2^{-4}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 64 & 1 & 1 \\ 1 & 64 & 1 & 1 \\ 1 & 64 & 1 & 1 \\ 1 & 64 & 1 & 1 \end{bmatrix}$$

Double-word algorithm: an example

$$A^{(0)}B^{(0)} + uA^{(1)}B^{(0)} =$$

$$\begin{bmatrix} 125 & 2^{-2} & 2^{-2} & 0 \\ 64 & 64 & 64 & 64 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 64 & 1 & 1 \\ 1 & 64 & 1 & 1 \\ 1 & 64 & 1 & 1 \\ 1 & 64 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & \mathbf{2^{-4}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 64 & 1 & 1 \\ 1 & 64 & 1 & 1 \\ 1 & 64 & 1 & 1 \\ 1 & 64 & 1 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 125.5 & 8032 & 125.5 & 125.5 \\ 256 & 16384 & 256 & 256 \\ 4 & 256 & 4 & 4 \\ 4 & 256 & 4 & 4 \end{bmatrix} + \begin{bmatrix} \mathbf{2^{-8}} & \mathbf{0.25} & \mathbf{2^{-8}} & \mathbf{2^{-8}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 125.50390625 & 8032.25 & 125.50390625 & 125.50390625 \\ 256 & 16384 & 256 & 256 \\ 4 & 256 & 4 & 4 \\ 4 & 256 & 4 & 4 \end{bmatrix}$$

Double-word algorithm: an example

$$C \approx \Lambda^{-1}\left(\sum_{i+j<p} u^{i+j} A^{(i)} B^{(j)}\right) M^{-1}.$$

Step 4: Undo the scaling.

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 125.50390625 & 8032.25 & 125.50390625 & 125.50390625 \\ 256 & 16384 & 256 & 256 \\ 4 & 256 & 4 & 4 \\ 4 & 256 & 4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 502.015625 & 64258 & 502.015625 & 502.015625 \\ 512 & 65536 & 512 & 512 \\ 4 & 512 & 4 & 4 \\ 4 & 512 & 4 & 4 \end{bmatrix} = AB.$$

Error analysis: summary

Single-word algorithm:

$$\|\widehat{C} - AB\|_\infty \lesssim \left(2u + nU + 4n^2\theta^{-1}g_{\min} + 4n^2\theta^{-2}G_{\min}\right)\|A\|_\infty\|B\|_\infty.$$

p-word algorithm:

$$\|\widehat{C} - AB\|_\infty \lesssim \Big((p+1)u^p + 4nu^{p-1}\theta^{-1}g_{\min} + (n+p^2)U + 2p(p+1)n^2\theta^{-2}G_{\min}\Big)\|A\|_\infty\|B\|_\infty.$$

Part 4: Numerical experiments

Numerical experiments

We generate $A \in \mathbb{R}^{10\times n}$ and $B \in \mathbb{R}^{n\times 10}$ and vary n.

Elements in $[-10^{10}, -10^{-10}] \cup [10^{-10}, 10^{10}]$.

Measure the accuracy with $\frac{\|\widehat{C}-C\|_\infty}{\|A\|_\infty \|B\|_\infty}$ where C is computed in binary64.

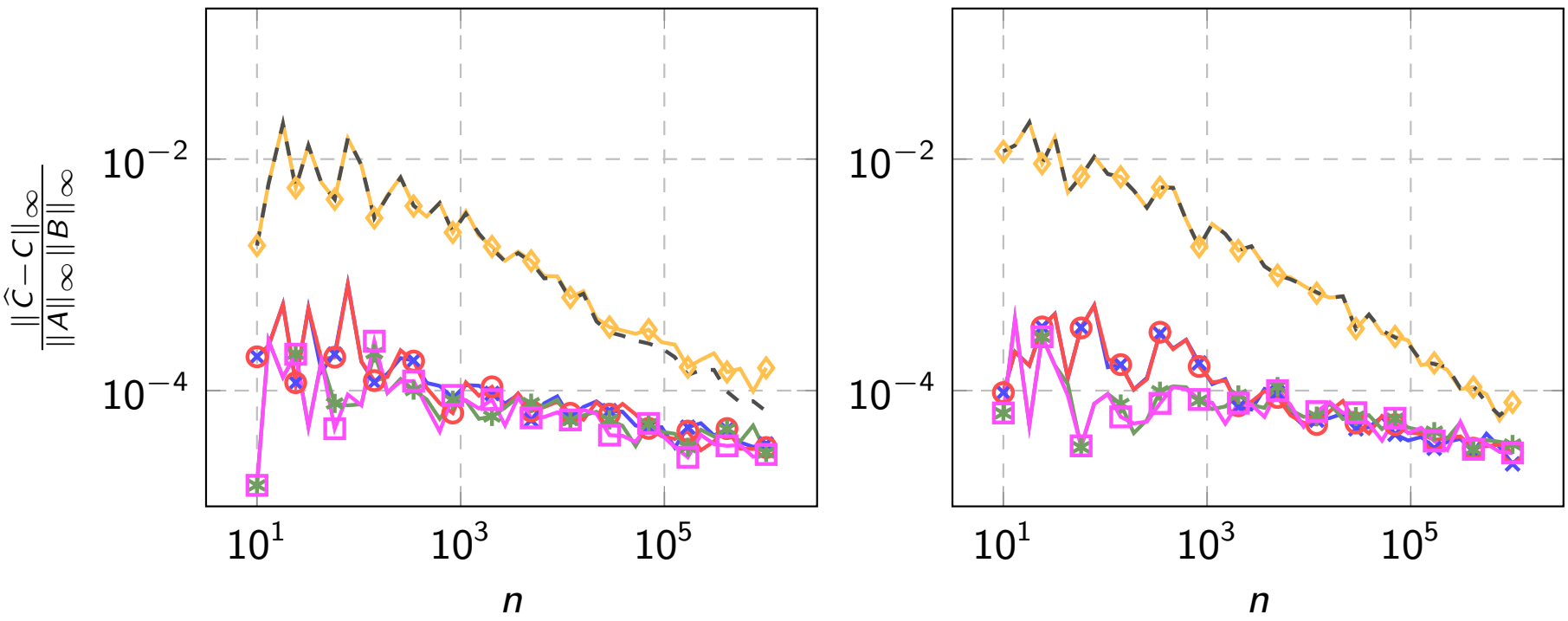
We check with subnormals on/off to detect any improvements due to *gradual underflow*.

We also plot the variants of MMA without any range (exponent) limitations.

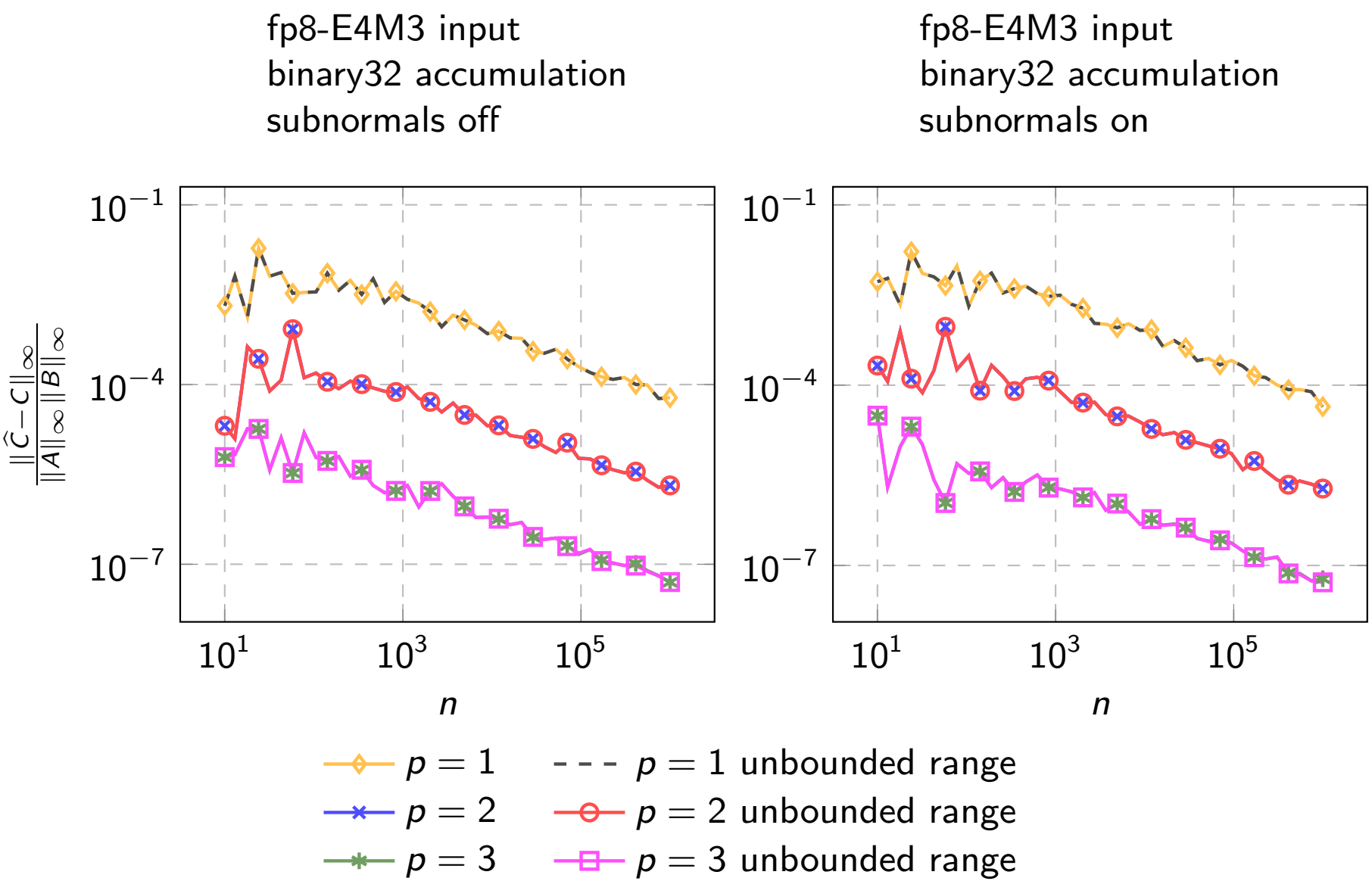
Numerical experiments I

fp8-E4M3 input
binary16 accumulation
subnormals off

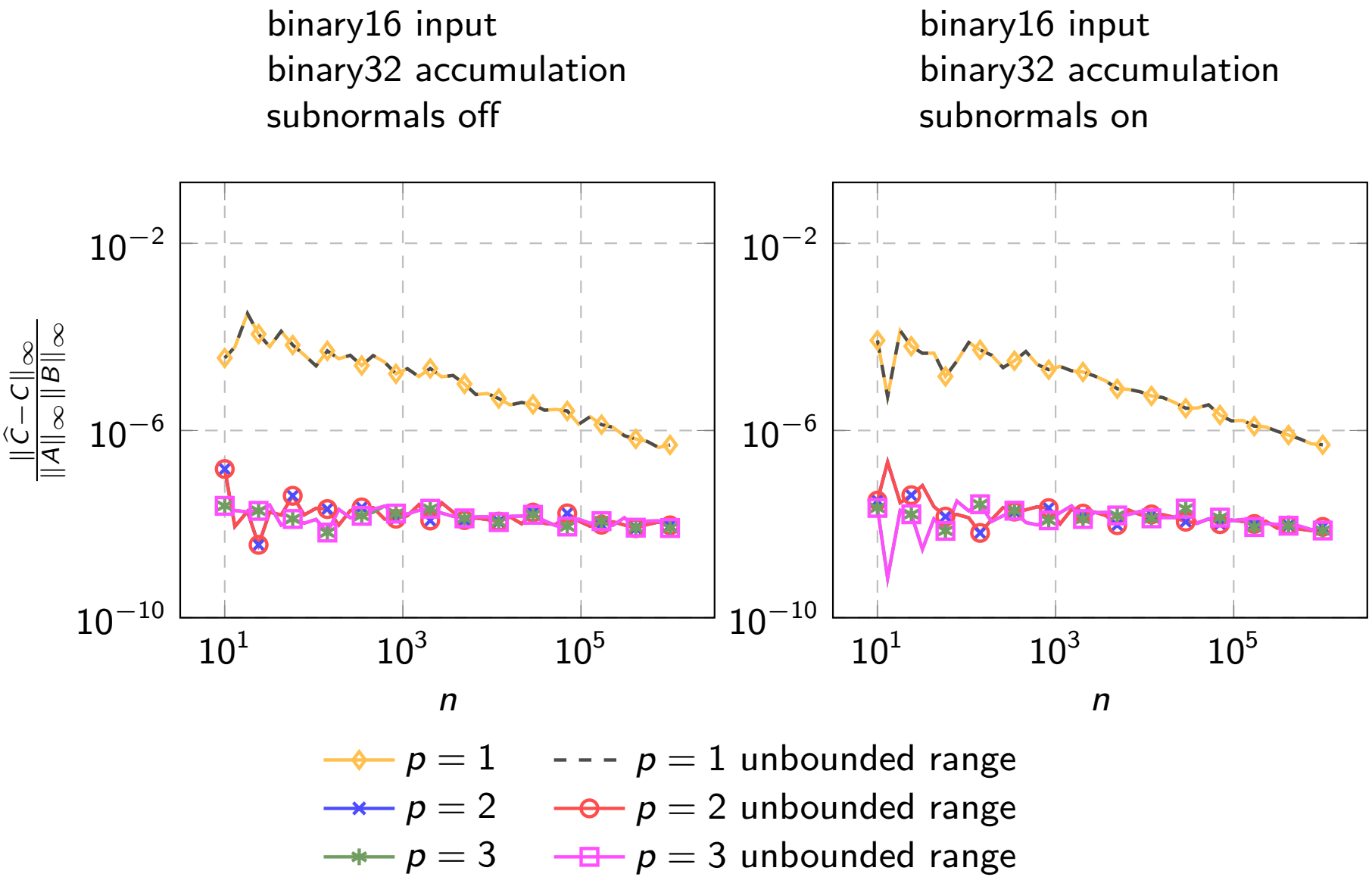
fp8-E4M3 input
binary16 accumulation
subnormals on



Numerical experiments II



Numerical experiments III



Summary

- Underflows in narrow-range FP formats not a problem, provided three types of scaling are used.
- Shown algorithms require minimal bit-level manipulations.
- Simple algorithms to obtain binary32 accuracy.
- Other algorithms exist combining MMA and binary64 addition.

SIAM SISC paper

T. Mary, and M. Mikaitis. *Error Analysis of Matrix Multiplication with Narrow Range Floating-Point Arithmetic*. **Preprint. Accepted for SIAM SISC**. Mar. 2025.

`https://bit.ly/42dqpkn`.

Slides at `http://mmikaitis.github.io/talks`

References I

J. Demmel
Underflow and the reliability of numerical software
SIAM J. Sci. Comput., 5:4. Dec. 1984.

P. Blanchard, N. J. Higham, F. Lopez, T. Mary, and S. Pranesh
Mixed precision block fused multiply-add: Error analysis and application to GPU tensor cores
SIAM J. Sci. Comput., 42:3. Jan. 2020.

M. Fasi, N. J. Higham, F. Lopez, T. Mary, and M. Mikaitis
Matrix Multiplication in Multiword Arithmetic: Error Analysis and Application to GPU Tensor Cores
SIAM J. Sci. Comput., 45:1. Feb. 2023.

M. Fasi and M. Mikaitis
CPFloat: A C library for Simulating Low-Precision Arithmetic
ACM Trans. Math. Software, 49. 2023